9 Complex integral

Here we finally get to the central point of our course — complex integral. Specifically, our task is make sense of the following notation

$$\int_\gamma f(z)\mathrm{d}z.$$

There are three parts of this formula: the integral itself, and what we actually understand by writing the integral sign; the curve or path γ; and finally the domain E, which should contain γ and on which f must be holomorphic (remember that the goal is to study holomorphic functions, not arbitrary ones). There are quite a few subtle geometric and topological points related to γ, E, and f, but I will try to get to the final result (Cauchy's theorem, see the next section) skipping most of these technical points. Full treatment should be looked for in a graduate textbook.

9.1 Path

By a path γ in complex analysis one usually calls a *rectifiable* curve. Loosely speaking, this is a continuous curve that has a finite length (but what is length anyway?). We will be satisfied with a somewhat much less strict definition of the path: Path γ is a continuous piecewise smooth function. This means that 1) $\gamma\colon [\alpha,\beta] \longrightarrow \mathbf{C}$ is continuous, i.e.,

$$\gamma(t) = x(t) + \mathrm{i}y(t),$$

and x and y are continuous real functions; moreover the derivative $\gamma'(t) = x'(t) + \mathrm{i}y'(t)$ exists at all the points of $[\alpha,\beta]$ except, possible, a finite number of points $t_1, t_2, \ldots, t_n$, and if derivative exists then it is a continuous function on its own.

Here are a few examples.

A path corresponding to the half circle through $1, \mathrm{i}, -1$ is given by $\gamma(t) = e^{\mathrm{i}t}$, $t \in [0,\pi]$.

A path corresponding to the line segment connecting points $x_0 + \mathrm{i}y_0$ and $x_1 + \mathrm{i}y_1$ is given by $\gamma(t) = (x_0 + (x_1 - x_0)t) + \mathrm{i}(y_0 + (y_1 - y_0)t), t \in [0,1]$, etc. More examples will be given below.

Quite often it is important to distinguish between the path itself (function $t \mapsto \gamma(t)$) and its image (the set of points $\{z \in \mathbf{C}\colon z = \gamma(t),\ t \in [\alpha,\beta]\}$), I will not do it in these lectures and will be using the same letter γ for both the path and its image.

Note that the path has a direction, I will denote $-\gamma$ the path that has the same image as γ but the opposite direction.

The path is called *closed* if $\gamma(\alpha) = \gamma(\beta)$; *simple* if it has no self intersections (except for possibly $\gamma(\alpha) = \gamma(\beta)$); a simple closed path is often called *a Jordan's curve.*

9.2 The integral

Now that we have some idea about paths I can define

Definition 9.1.

$$\int_\gamma f(z)\mathrm{d}z = \int_\alpha^\beta f(\gamma(t))\gamma'(t)\mathrm{d}t,$$

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where on the right is a pair of usual familiar real integrals

$$\int_\alpha^\beta f(\gamma(t))\gamma'(t)\mathrm{d}t = \int_\alpha^\beta \mathrm{Re}(f(\gamma(t))\gamma'(t))\mathrm{d}t + \mathrm{i}\int_\alpha^\beta \mathrm{Im}(f(\gamma(t))\gamma'(t))\mathrm{d}t.$$

Note that this definition of complex integral is perfectly fine even for the paths that are not differentiable at every point, because the Riemann integral exists for piecewise continuous functions, and this is what I required from γ' to satisfy.

The definition may look a little arbitrary, and in this respect I would like to remark that $\int_\gamma f$ is nothing else as two real line integrals, which were studied in Calc III, and which have quite natural geometric and physical interpretations. Indeed, assuming $f = u + \mathrm{i}v$ and $z = x + \mathrm{i}y$,

$$\begin{aligned}\int_\gamma f(z)\mathrm{d}z &= \int_\alpha^\beta (u+\mathrm{i}v)(x'+\mathrm{i}y')\mathrm{d}t = \\ &= \int_\alpha^\beta (ux'-vy')\mathrm{d}t + \mathrm{i}\int_\alpha^\beta (vx'+uy')\mathrm{d}t = \\ &= \int_\gamma u\mathrm{d}x - v\mathrm{d}y + \mathrm{i}\int_\gamma v\mathrm{d}x + u\mathrm{d}y.\end{aligned}$$

Here is our first (and arguably most important) example.

Example 9.2. Let $\gamma = \partial B(0,1)$, i.e., the boundary of the unit disk, and $n \in \mathbf{Z}$. Then

$$\int_\gamma z^n = \begin{cases} 0, & n \neq -1, \\ 2\pi\mathrm{i}, & n = -1. \end{cases}$$

In this case I have $\gamma(t) = e^{\mathrm{i}t}, t \in [0, 2\pi]$, hence

$$\int_\gamma z^n \mathrm{d}z = \mathrm{i}\int_0^{2\pi} e^{it(n+1)}\mathrm{d}t,$$

and the result follows from $\int e^{at}\mathrm{d}t = \frac{1}{a}e^{at}$ for any complex a.

Here are a few technical points which will be important in the following. The proofs rely on the properties of real integrals, which I assume to be known.

1. $$\int_\gamma f = \int_{-\gamma} f.$$

   The proof follows from $\int_a^b = -\int_b^a$.

2. Let $\gamma = \gamma_1 \cup \gamma_2$ and γ_1 and γ_2 are disjoint. Then

   $$\int_\gamma f = \int_{\gamma_1} f + \int_{\gamma_2} f.$$

   Clearly one can generalize to any finite number of γ's. Proof follows from the definition and properties of real integrals.

3. *Reparametrization.* This is one of the most important properties which informally says that the complex integral depends only on the image of γ and its direction, but not on a concrete parametrization of this image. To be specific: Let $\tilde{\gamma}\colon[\tilde{\alpha},\tilde{\beta}]\longrightarrow\mathbf{C}$ be another path, and assume that $\tilde{\gamma}=\gamma\circ\psi$, where function $\psi\colon[\tilde{\alpha},\tilde{\beta}]\longrightarrow[\alpha,\beta]$, onto, and has a positive derivative. Then

$$\int_{\tilde{\gamma}} f=\int_{\gamma} f.$$

Proof. Due to Property 2 I can concentrate on an interval where both γ' and $\tilde{\gamma}'$ exist. Then

$$\tilde{\gamma}'(t)=\gamma'(\psi(t))\psi'(t)$$

by the chain rule.

$$\begin{aligned}\int_{\tilde{\gamma}} f(z)\mathrm{d}z&=\int_{\tilde{\alpha}}^{\tilde{\beta}} f(\tilde{\gamma}(t))\tilde{\gamma}'(t)\mathrm{d}t=\\&=\int_{\tilde{\alpha}}^{\tilde{\beta}} f(\gamma(\psi(t)))\gamma'(\psi(t))\psi'(t)\mathrm{d}t=\text{(changing the variables)}=\\&=\int_{\alpha}^{\beta} f(\gamma(s))\gamma(s)\mathrm{d}s=\int_{\gamma} f(z)\mathrm{d}z.\end{aligned}$$

■

Property 3 is especially important because it allows to choose the simplest paramentrization of our path for each smooth segment. Here is an example.

Example 9.3. Find

$$\int_{\gamma} z^2\mathrm{d}z,$$

where $\gamma=\gamma_1\cup\gamma_2$, γ_1 is the line segment of the real axis from $-R$ to R and γ_2 is the half circle in the upper half plane connecting points R and $-R$. Due to property 2 I can separate this integrals into two: along γ_1 and γ_2 correspondingly. I also have

$$\gamma_1(t)=-R+2Rt,\quad t\in[0,1],$$

and

$$\gamma_2(t)=Re^{it},\quad t\in[0,\pi],$$

note that my t is not continuous along the whole γ because by Property 3 I can choose the parametrization that I like. Finally,

$$\begin{aligned}\int_{\gamma} z^2\mathrm{d}z&=\int_0^1 R^2(2t-1)^2 2R\mathrm{d}t+\int_0^{\pi} R^2e^{2\mathrm{i}t}\mathrm{i}Re^{\mathrm{i}t}\mathrm{d}t=\\&=\frac{2R^3}{2}\frac{(2t-1)^3}{3}\Big|_0^1+\frac{R^3\mathrm{i}}{3}e^{3\mathrm{i}t}\Big|_0^{\pi}=\\&=\frac{1}{3}R^3+\frac{1}{3}R^3+\frac{R^3}{3}(e^{3\pi\mathrm{i}}-1)=0.\end{aligned}$$

The calculations in the last examples are nice to emphasize the importance of the properties of complex integrals, but, strictly speaking, are not necessary. In the next subsection I will explain the reasons for this.

9.3 Antiderivative

Judging by its title the Fundamental theorem of calculus is the central theorem in Calc I. Here I will show how to formulate a complex analogue of this theorem, but would like to remark from the very beginning that this "fundamental theorem" is a particular case of a much more general statement (Cauchy's theorem) and hence plays only an axillary role.

I start with the usual definition that complex function F is called an antiderivative of f in domain E if $F'(z) = f(z)$ for all $z \in E$. From our experience we have that, e.g., z^2 is an antiderivative for $2z$ in all $\mathbf{C}$, z^{-1} is an antiderivative of $-1/z^2$ in $\mathbf{C} \setminus \{0\}$, and $\operatorname{Log} z$ is an antiderivative of $1/z$ in E, where E has no points on the negative real half-axis. Note that similar to the real case antiderivative is not unique, but is determined (on a domain E) up to an additive complex constant. I have

Proposition 9.4. *Let F be an antiderivative of f on domain E, and let $\gamma \in E$. Then*

$$\int_\gamma f(z)\mathrm{d}z = F(\gamma(\beta)) - F(\gamma(\alpha)).$$

Proof. I will assume that γ is smooth. The full version is obtained by invoking Property 2 from the previous section.

$$\begin{aligned}\int_\gamma f(z)\mathrm{d}z &= \int_\alpha^\beta F'(\gamma(t))\gamma'(t)\mathrm{d}t = (s = \gamma(t)) = \\ &= \int_{\gamma(\alpha)}^{\gamma(\beta)} F'(s)\mathrm{d}s = F(\gamma(\beta)) - F(\gamma(\alpha))\end{aligned}$$

as required. ∎

And here is our first glimpse of Cauchy's theorem as an immediate corollary to the previous.

Corollary 9.5. *Let γ be a closed path, $\gamma \in E$, and f has an antiderivative F in E. Then*

$$\oint_\gamma f(z)\mathrm{d}z = 0.$$

(Here I use the notation $\oint$ to emphasize that the integral is taken along a closed path.)

To use this corollary one needs to be sure that an antiderivative exists, which is not always clear. The contrapositive, however, gives a direct recipe to prove that there is no antiderivative in some domain E. For instance we know (see the fundamental example above) that

$$\int_{\partial B(0,1)} \frac{1}{z}\mathrm{d}z = 2\pi\mathrm{i},$$

which implies that $\frac{1}{z}$ has no antiderivative in any domain containing the unit circle.

Finally since z^2 is entire, and clearly has antiderivative $z^3/3$ in the whole $\mathbf{C}$, then the calculations in Example 9.3 were not necessary since we integrated along a closed path.

9.4 One useful estimate

For real integrals we have (which is in a sense a generalized triangle inequality)

$$\left|\int_a^b f(x)\mathrm{d}x\right| \leq \int_a^b |f(x)|\mathrm{d}x.$$

The proof follows from the inequalities $f(x) \leq |f(x)|$ and $-f(x) \leq |f(x)|$ and direct integration. One cannot assume that the same inequality holds for complex integrals. Indeed, assume that something similar holds

$$\left|\int_{\partial B(0,1)} \frac{1}{z}\mathrm{d}z\right| \leq \int_{\partial B(0,1)} \frac{1}{|z|}\mathrm{d}z = \int_0^{2\pi} \mathrm{i}e^{\mathrm{i}t}\mathrm{d}t = 0,$$

which does not make any sense since we already know that this integral is equal $2\pi\mathrm{i}$. Here is the correct way to estimate complex integrals.

Proposition 9.6. *Let E be a domain, $\gamma \in E$, and $M = \max_{z\in\gamma} |f(z)|$. Then*

$$\left|\int_\gamma f(z)\mathrm{d}z\right| \leq M \int_\alpha^\beta |\gamma'(t)|\mathrm{d}t = M \cdot \operatorname{length}\gamma,$$

where I defined

$$\operatorname{length}\gamma = \int_\alpha^\beta |\gamma'(t)|\mathrm{d}t.$$

Proof. The proof is not very illuminating and a little technical, but it uses one trick, which is very useful to remember while working with complex numbers. So I provide the details. I need to estimate $\left|\int_\gamma f\right|$. I will use the fact that

$$\int_\gamma f = \left|\int_\gamma f\right| e^{\mathrm{i}\theta},$$

where θ is the argument of $\int_\gamma f$. Therefore,

$$\left|\int_\gamma f\right| = e^{-\mathrm{i}\theta} \int_\gamma f.$$

Now, the expression on the left is real, hence the expression on the right must be real and hence coincides with its real part:

$$\left|\int_\gamma f\right| = \operatorname{Re} \int_\gamma e^{-\mathrm{i}\theta} f(z)\mathrm{d}z = \int_\alpha^\beta \operatorname{Re}\left[f(\gamma(t))e^{\mathrm{i}\theta}\gamma'(t)\right]\mathrm{d}t.$$

Now, the right hand side is real, and I can use the integral inequality for the real integrals

$$\left|\int_\gamma f\right| \leq \int_\alpha^\beta \left|f(\gamma(t))e^{\mathrm{i}\theta}\gamma'(t)\right|\mathrm{d}t = \int_\alpha^\beta |f(\gamma(t))||\gamma'(t)|\mathrm{d}t,$$

from where the required inequality follows. ■

Example 9.7. Let γ be the semicircle in the upper half plane of radius R and $f(z) = (z^4+1)^{-1}$. I have

$$\left|\int_\gamma f(z)\mathrm{d}z\right| \leq \int_0^\pi \left|\frac{R\mathrm{i}e^{\mathrm{i}t}}{R^4e^{4\mathrm{i}t}+1}\right|\mathrm{d}t \leq \frac{R}{|R^4-1|}\pi,$$

where I also used the inverse triangle inequality $|a-b| \geq ||a|-|b||$.